TCC Week 2.

AAP principle. Assume $u_* > 0$ is a solution to

$$-\Delta u + Vu = f \text{ in } \Omega \subseteq \mathbb{R}^N$$

$$\Rightarrow \int |\nabla \varphi|^2 + \int V \varphi^2 = \int \underbrace{\left|\nabla \frac{\varphi}{u_*}\right|^2 u_*^2}_{\geq 0} + \int \frac{f}{u_*} \varphi^2 ,$$

$$\forall \varphi \in C_0^\infty(\Omega)$$

$$\Rightarrow \int_{\mathbb{R}^N} |\nabla \varphi|^2 \geq C_H \int \frac{\varphi^2}{|x|^2} , \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$$

$$C_H = \left(\frac{N-2}{2}\right)^2 , \quad N \geq 3$$

To prove H.I. simply take $u_* = |x|^{-\frac{N-2}{2}}$:

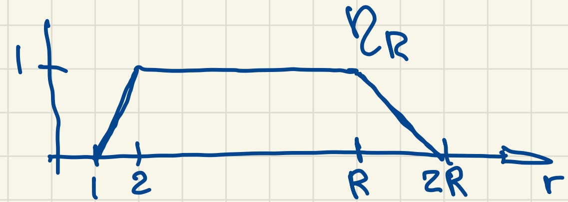
$$-\Delta u_* - \frac{C_H}{|x|^2} u_* = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}$$

Why C_H is optimal? Assume not:

$$\int |\nabla \varphi|^2 - (C_H + \varepsilon) \int \frac{\varphi^2}{|x|^2} \geq 0 \;?$$

Take $u_* = |x|^{-\frac{N-2}{2}}$, $\underbrace{\eta_R}_{\text{cut off}}$

$\varphi_R = \eta_R u_*$



$$\int |\nabla \varphi_R|^2 - (C_H+\varepsilon) \int \frac{\varphi_R^2}{|x|^2} \underset{R\to\infty}{=} \int \underbrace{|\nabla \eta_R|^2}_{\leq C_1} u_*^2 - \varepsilon \underbrace{\int \frac{\varphi_R^2}{|x|^2}}_{\to +\infty} < 0$$

$\varphi_R \to u_*$ $(|x|>2)$, $\int \frac{u_*^2}{|x|^2} = +\infty$ $\xrightarrow{R\to\infty}$

$\Rightarrow C_H$ is optimal

Improved Hardy inequality

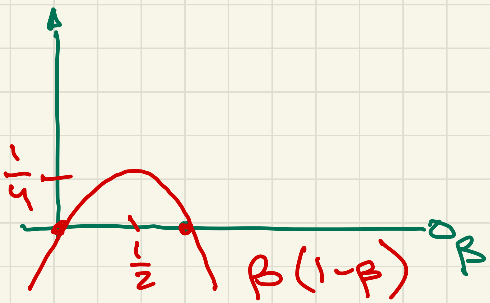
$$u(r) = r^{-\frac{N-2}{2}} \log^{\beta}(r) > 0 \quad \text{in } |x|>1$$

$$-\Delta u - \frac{C_H}{|x|^2} u = \beta(1-\beta)\, \underbrace{r^{-\frac{N-2}{2}-2}}_{}\log^{\beta-2}(r)$$

$$= \frac{\beta(1-\beta)}{|x|^2 \log^2|x|}\, u$$

Divide by u!

$$\int |\nabla \varphi|^2 - C_H \int \frac{\varphi^2}{|x|^2} \overset{\text{AAP}}{\geq} \underbrace{\beta(1-\beta)}_{\frac{1}{4} \text{ iff } \beta=\frac{1}{2}} \int \frac{\varphi^2}{|x|^2 \log^2|x|}, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N \setminus \bar{B}_1)$$



Improved Hardy Ineq:

$$\int |\nabla \varphi|^2 \geq C_H \int \frac{\varphi^2}{|x|^2} + \frac{1}{4} \int \frac{\varphi^2}{|x|^2 \log^2 \left|\frac{x}{R}\right|},$$

$$u_* = |x|^{-\frac{N-2}{2}} \log^{\frac{1}{2}} \left|\frac{x}{R}\right| \qquad \forall \varphi \in C^\infty(\mathbb{R}^N \setminus B_R)$$

First Dirichlet eigenvalue in $\Omega \subseteq \mathbb{R}^N$

$$\begin{cases} -\Delta \varphi_1 = \lambda_1 \varphi_1 & \text{in } \Omega \subset \mathbb{R}^N \\ \varphi_1 = 0 & \text{on } \partial\Omega \end{cases}$$ – bounded domain

$\varphi_1 > 0$ in Ω – principal eigenvalue

$\lambda_1 > 0$ – principal eigenfunction

AAP $\Longrightarrow$ $\displaystyle\int_\Omega |\nabla \varphi|^2 \geq \lambda_1 \int_\Omega \varphi^2 \quad \forall\, \varphi \in C_0^\infty(\Omega)$

$\blacktriangleleft$ $-\Delta\varphi_1 - \lambda_1\varphi_1 = 0$, $\varphi_1 > 0$. Take $u_* = \varphi_1$ $\blacktriangleright$

Torsional Hardy inequality

$$\begin{cases} -\Delta \psi_\Omega = 1 & \text{in } \Omega \subseteq \mathbb{R}^N \\ \psi_\Omega = 0 & \text{on } \partial\Omega \end{cases}$$

– bounded domain

$\psi_\Omega \in C^2(\overline{\Omega})$

– torsion function of Ω, $\psi_\Omega > 0$

AAP $\implies$

$$\int |\nabla \varphi|^2 \geq \int \frac{1}{\psi_\Omega} \varphi^2 \quad \forall \varphi \in C_0^\infty(\Omega)$$

– torsional Hardy inequality

Exercise: $\longrightarrow$

Exercise: $\|\psi_\Omega\|_\infty \geq \frac{1}{\lambda_1}$

φ_1 – principal eigenfunct, $\int \varphi_1^2 = 1$

$$\lambda_1 \underbrace{\int \varphi_1^2}_{=1} = \int |\nabla \varphi_1|^2 \geq \int \frac{1}{\psi_\Omega} \varphi_1^2 \geq \inf \frac{1}{\psi_\Omega} \underbrace{\int \varphi_1^2}_{=1}$$

$$= \frac{1}{\|\psi_\Omega\|_\infty}$$

$$\Omega = B_R \Rightarrow \lambda_1 \simeq \frac{c}{R^2}$$

$$\Rightarrow \|\psi_{B_R}\|_\infty = \psi_{B_R}(0) \geq \frac{1}{\lambda_1} = \frac{c}{R^2}$$

2. Energy space

AAP principle: Assume $\exists\, u_* > 0$:

$$-\Delta u_* + V u_* \geq f \text{ in } \Omega, \quad f \geq 0$$

Then $E_V(\varphi) := \int_\Omega |\nabla\varphi|^2 + V\varphi^2 \geq \int \frac{f}{u_*}\varphi^2$,

$\forall \varphi \in C_0^\infty(\Omega)$

Note that $\|\varphi\|_V = \left(E_V(\varphi)\right)^{1/2}$ is a "norm"

$\langle \varphi, \psi \rangle_V = \int \nabla\varphi \nabla\psi + \int V\varphi\psi$ is a "scalar product"

If $V \geq 0$ this is easy to see

Theorem. Assume $\exists\, \alpha \in L^1_{loc}(\Omega)$, $\alpha > 0$ a.e.

$$E_*(\varphi) \geq \int \underbrace{\alpha(x)}_{f/u_*} \varphi^2 \quad \forall \varphi \in C_0^\infty(\Omega) \qquad (*)$$

Then the completion of $C_0^\infty(\Omega)$ w.r.t.

$\|\varphi\|_V = \left(E_*(\varphi)\right)^{1/2}$ is the Hilbert space $\mathcal{D}^1_V(\Omega)$

$\langle \varphi, \psi \rangle_V = \int \nabla\varphi \nabla\psi + \int V \varphi \psi$ – scalar prod.

$$\mathcal{D}^1_V(\Omega) \hookrightarrow L^2(\Omega, \alpha(x)\,dx)$$

$(*)$ – Lambda property

Sketch of the proof:

1) $\|\varphi\|_V$ is a norm on $C_0^\infty(\Omega)$ $\Big(\underbrace{\|\varphi\|_V = 0 \Leftrightarrow \varphi = 0}_{\text{Because } \alpha > 0}\Big)$

2) (φ_n) — Cauchy sequence w.r.t. $\|\varphi\|_V$

$\Rightarrow$ (φ_n) is Cauchy in $L^2(\Omega, \alpha(x)dx)$

$$\Big(\int \underbrace{|\varphi_n - \varphi_m|^2}_{\to 0} \alpha(x)\Big)^{1/2} \underset{*}{\leq} \big(E(\varphi_n - \varphi_m)\big)^{1/2} = \underbrace{\|\varphi_n - \varphi_m\|_V}_{\to 0}$$

$\Rightarrow$ $\varphi_n \xrightarrow[n\to\infty]{L^2} \varphi_0$ (L^2 is Banach)

Define $E_v(\varphi_0) := \lim_{n\to\infty} E(\varphi_n)$

We say then $\varphi_0 \in \mathcal{D}_v^1(\Omega)$ – the completion of $C_0^\infty(\Omega)$ w.r.t. to $\|\cdot\|_v$

We can do this for every Cauchy seq.

Moreover, $\mathcal{D}_v^1(\Omega) \hookrightarrow L^2(\Omega, \lambda\, dx)$ ▶

Examplee: $-\Delta$ on $\mathbb{R}^N$, $N \geq 3$.

Take $u_* = (1+|x|^2)^{-\frac{N-2}{2}}$ – Talenti function

$$-\Delta u_* = c_1 \underbrace{(1+|x|^2)^{-\frac{N+2}{2}}}_{f} \simeq u_*^{\frac{N+2}{N-2}} \text{ in } \mathbb{R}^N$$

$$a(x) = \frac{f}{u_*} = \frac{u_*^{\frac{N+2}{N-2}}}{u_*} = u_*^{\frac{4}{N-2}} \simeq (1+|x|^2)^{-1}$$

$$\Rightarrow \mathcal{D}^1_0(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N, (1+|x|^2)^{-1}dx)$$

$\mathcal{D}^1_0(\mathbb{R}^N)$ – completion of $C_0^\infty(\mathbb{R}^N)$

w.r.t. $\left(\int |\nabla \varphi|^2\right)^{1/2}$ – homogeneous Sobolev

norm

$H^1(\mathbb{R}^N)$ – completion of $C^\infty(\mathbb{R}^N)$ w.r.t. $\left(\int |\nabla \varphi|^2 + \int \varphi^2\right)^{1/2}$
– well defined for $N \geq 1$.

$\mathcal{D}^1_0(\mathbb{R}^N)$ – defined for $N \geq 3$ only

$\mathcal{D}^1_0(\mathbb{R}^N) \neq H^1(\mathbb{R}^N)$

$-\Delta$ on $\mathbb{R}^2$ does not satisfy a-property

Liouville thm. – const is the only positive superharmonic on $\mathbb{R}^2$